15UEC904 – LINEAR CONTROL ENGINEERING

REGULATION - 2015

UNIT II - TIME RESPONSE ANALYSIS

CO.2	**Analyze the time response of first and Second order systems. (K4 - Analyze)**

Time response analysis - First Order Systems - Impulse and Step Response analysis of second order systems - Steady state errors – P, PI, PD and PID Compensation, Analysis using MATLAB (9)

Topics to be covered

- Time response
- Types of test input
- I order/II order system responses
- Time domain specification
- Error Coefficients
- Steady state errors
- P. PI, PD, PID controllers

Introduction

- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.

- Time response of the system is defined as the output of a system when subjected to an input which is a function of time.
- It is possible to compute the **time response of a system** if the **nature of input** and the **mathematical model of the system** are known.
- A control system generates an output or response for given input.
- The input represents the desired response while the output is actual response of system.

DEFINITIONS

TIME RESPONSE: The time response of a system is the output (response) which is function of the time, when input (excitation) is applied.

Time response of a control system consists of two parts

1. Transient Response

2. Steady State Response

Mathematically, $c(t) = c_t(t) + c_{ss}(t)$

Where, $c_t(t)$ = transient response

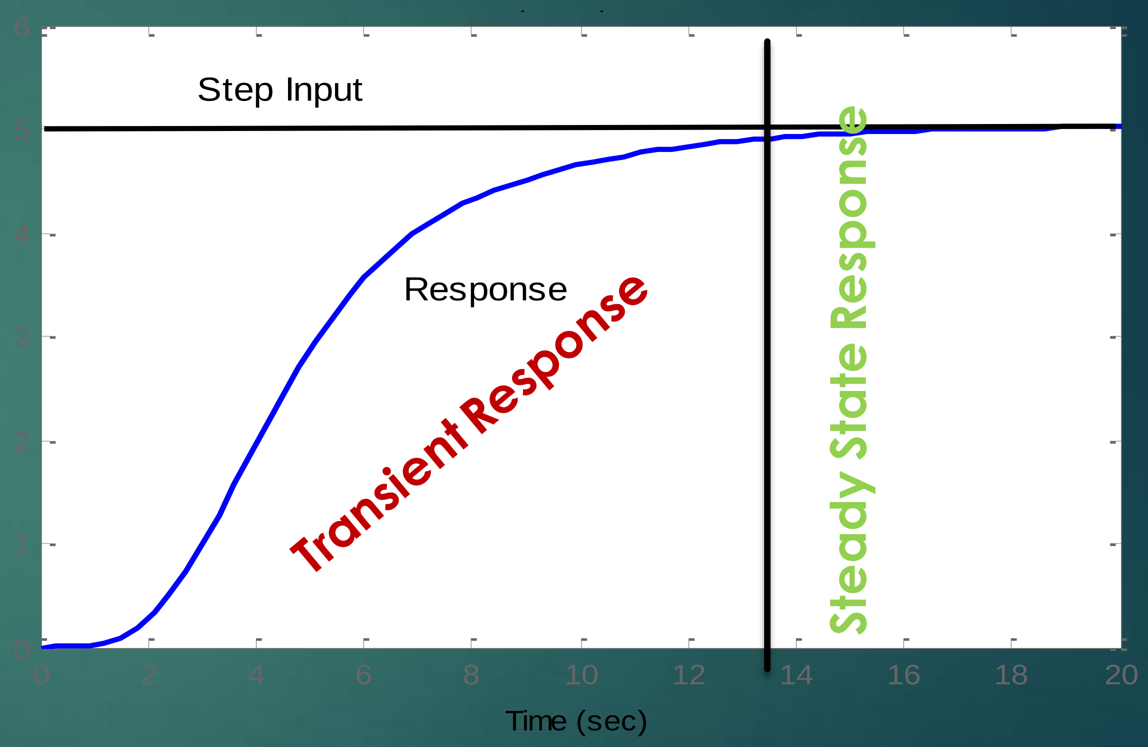
$c_{ss}(t)$ = steady state response

Time Response of Control Systems

- When the response of the system is changed from equilibrium it takes some time to settle down.

- This is called transient response.

- The response of the system after the transient response is called steady state response.



Time Response of Control Systems

- Transient response depend upon the system poles only and not on the type of input.
- It is therefore sufficient to analyze the transient response using a step input.
- The steady-state response depends on system dynamics and the input quantity.
- It is then examined by using different test signals.

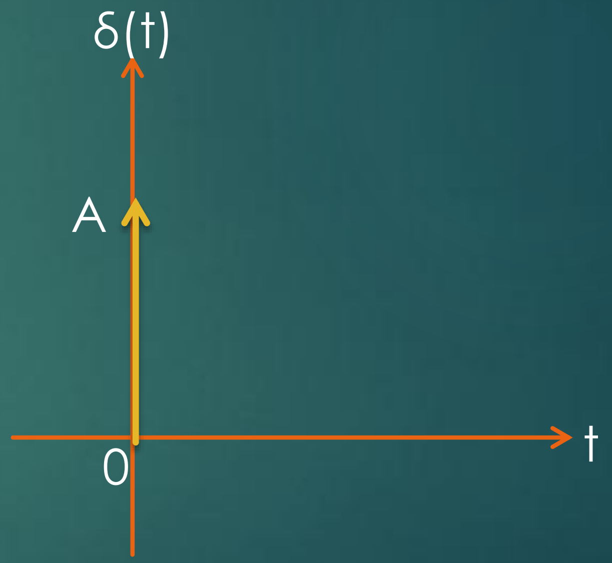
Standard Test Signals

- For time response analysis of control systems, we need to subject the system to various test inputs.
- Test input signals are used for testing how well a system responds to known input.
- The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration.
- The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.
- The other standard signal of great importance is a sinusoidal signal.

Standard Test Signals

- **Impulse signal**
  - The impulse signal imitate the sudden shock characteristic of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

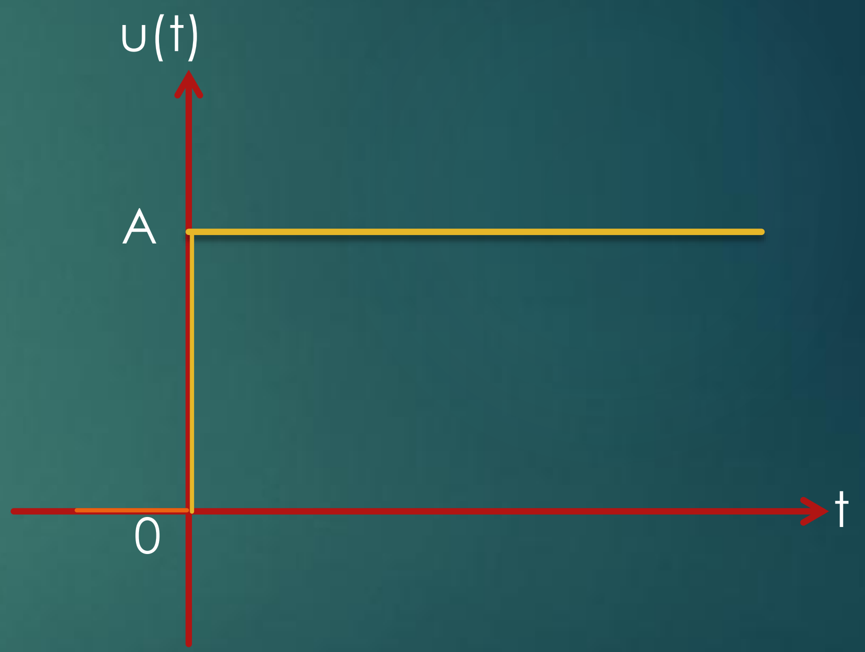


  - If A=1, the impulse signal is called unit impulse signal.

Standard Test Signals

- **Step signal**
  - The step signal imitate the sudden change characteristic of actual input signal.

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$



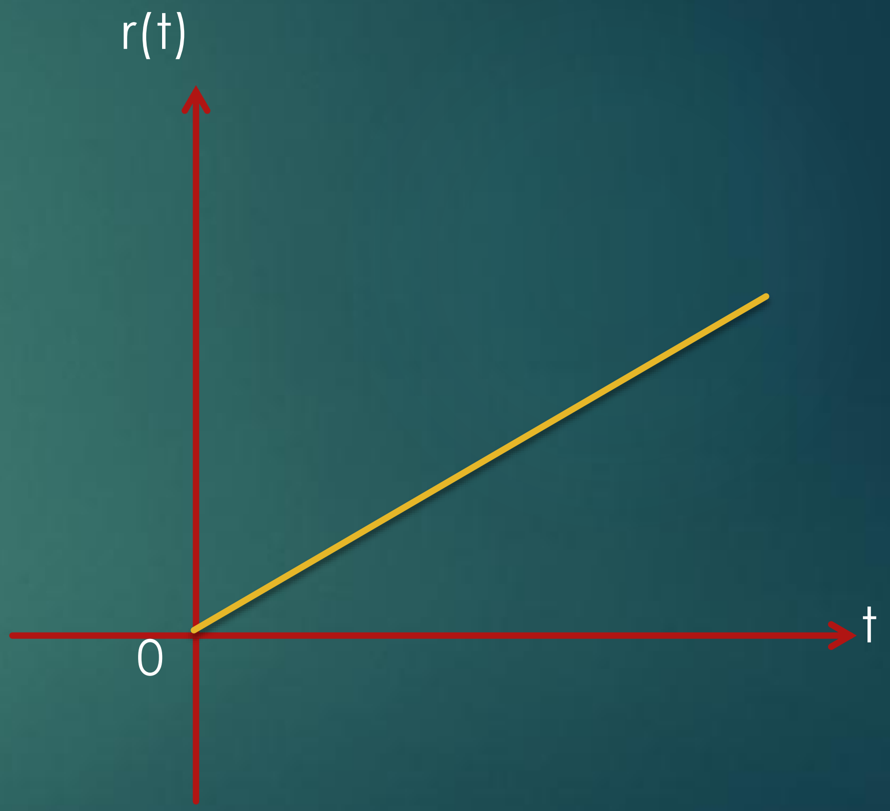
  - If A=1, the step signal is called unit step signal

Standard Test Signals

- **Ramp signal**
  - The ramp signal imitate the constant velocity characteristic of actual input signal.

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

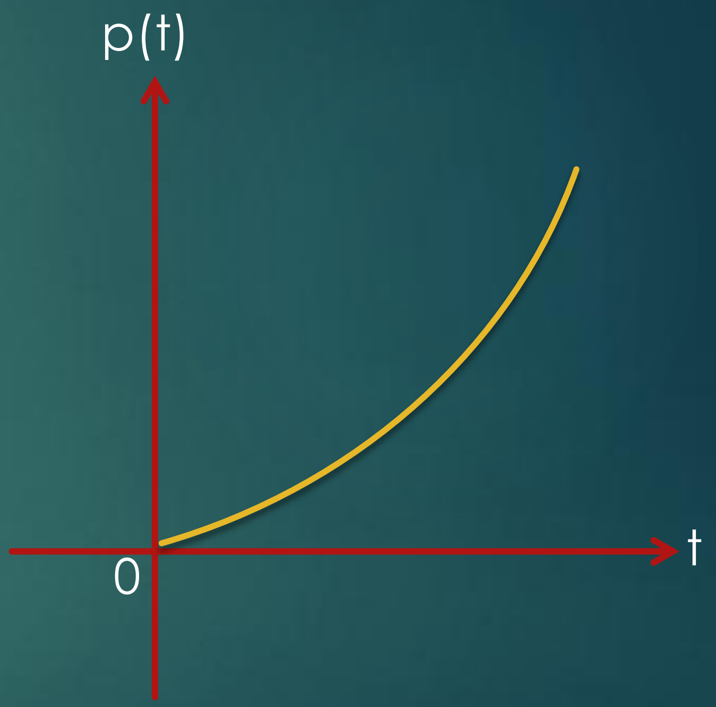
  - If *A=1*, the ramp signal is called unit ramp signal



Standard Test Signals

- **Parabolic signal**
  - The parabolic signal imitate the constant acceleration characteristic of actual input signal.

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



  - If $A=1$, the parabolic signal is called unit parabolic signal.

Relation between standard Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Parabolic

$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Integration ($\int$) leads from each signal to the next: impulse → step → ramp → parabolic.

Differentiation ($\frac{d}{dt}$) leads from each signal to the previous: parabolic → ramp → step → impulse.

TABLE 2-1 : Standard Test Signals

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	$\frac{A}{s}$
Unit step	1	$\frac{1}{s}$
Ramp	At	$\frac{A}{s^2}$
Unit ramp	t	$\frac{1}{s^2}$
Parabolic	$\frac{At^2}{2}$	$\frac{A}{s^3}$
Unit parabolic	$\frac{t^2}{2}$	$\frac{1}{s^3}$
Impulse	$\delta(t)$	1

ORDER OF THE SYSTEMS

$$\text{Transfer function, } T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \ldots\ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots + a_{n-1} s + a_n}$$

where, $P(s)$ = Numerator polynomial

$Q(s)$ = Denominator polynomial

Now, **n** is the order of the system

When $n = 0$, the system is zero order system.

When $n = 1$, the system is first order system.

When $n = 2$, the system is second order system and so on.

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\ldots\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots\ldots(s+p_n)}$$

where, $z_1, z_2, \ldots\ldots z_m$ are zeros of the system.

$p_1, p_2, \ldots\ldots p_n$ are poles of the system.

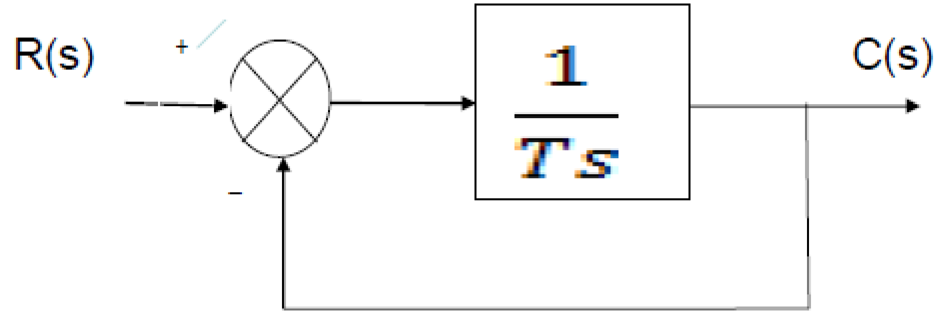
The value of ***n*** gives the **no. of poles** in the transfer function.

Hence the **order** is also given by the **no. of poles** of the transfer function.

TIME RESPONSE OF FIRST ORDER SYSTEMS

For Impulse Input:

The response of the system when the input signal is an Impulse signal.



$$R(s) = \delta(s) = 1$$

The transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

- Where ***K*** is the D.C gain and ***T*** is the time constant of the system.

- Time constant is a measure of how quickly a 1st order system responds to a unit step input.

- D.C Gain of the system is ratio between the input signal and the steady state value of output.

Impulse Response of 1st Order System

$$C(s) = \frac{K}{Ts+1}$$

- Re-arrange following equation as

$$C(s) = \frac{K/T}{s+1/T}$$

$$c(t) = \frac{K}{T}e^{-t/T}$$

- In order to compute the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

$$L^{-1}\left(\frac{C}{s+a}\right) = Ce^{-at}$$

- **The first order system given below.**

$$G(s)=\frac{10}{3s+1}$$

- D.C gain is **10** and time constant is **3 seconds**.

- For the following system

$$G(s)=\frac{3}{s+5}$$

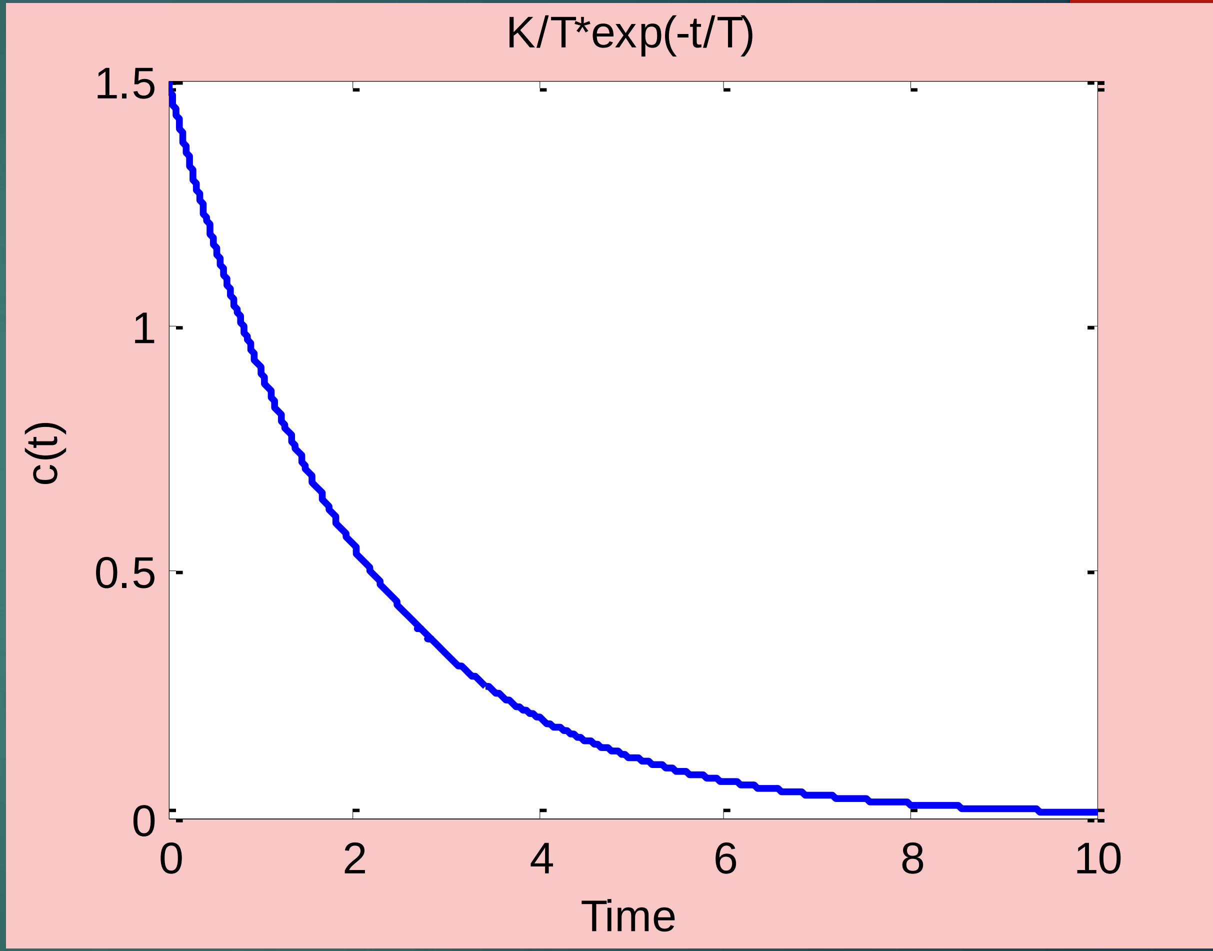
$$=\frac{3/5}{1/5s+1}$$

- D.C Gain of the system is **3/5** and time constant is **1/5 seconds.**

Impulse Response of 1st Order System

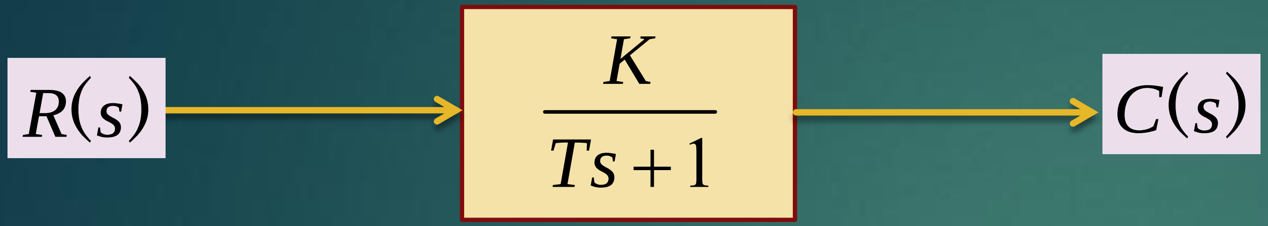
- If K=3 and T=2s then

$$c(t) = \frac{K}{T} e^{-t/T}$$



Step Response of 1st Order System

- Consider the following 1st order system with step input



$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1} = \frac{1}{Ts+1}; \; Let\ K = 1$$

For step input;

$$r(t) = u(t) = 1; \; t >= 0$$

$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{R(s)}{(Ts+1)} = \frac{1/s}{Ts+1} = \frac{1}{s(Ts+1)}$$

Step Response of 1st Order System

$$C(s) = \frac{R(s)}{(Ts+1)} = \frac{1/s}{Ts+1} = \frac{1/T}{s(s+1/T)}$$

$$put\ s = 0;\quad A = 1$$

$$put\ s = -1/T;\ B = -1$$

$$C(s) = \frac{1/T}{s\left(s+\frac{1}{T}\right)} = \frac{A}{s} + \frac{B}{s+\frac{1}{T}}$$

$$1/T = A\left(s+\frac{1}{T}\right) + Bs$$

$$C(s) = \frac{1}{s} + \frac{-1}{s+\frac{1}{T}}$$

Step Response of 1st Order System

- Taking Inverse Laplace of this equation

$$C(s) = \frac{1}{s} + \frac{-1}{s + \frac{1}{T}}$$

$$c(t) = \left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

$$c(t) = \left(1 - e^{-t/T}\right)$$

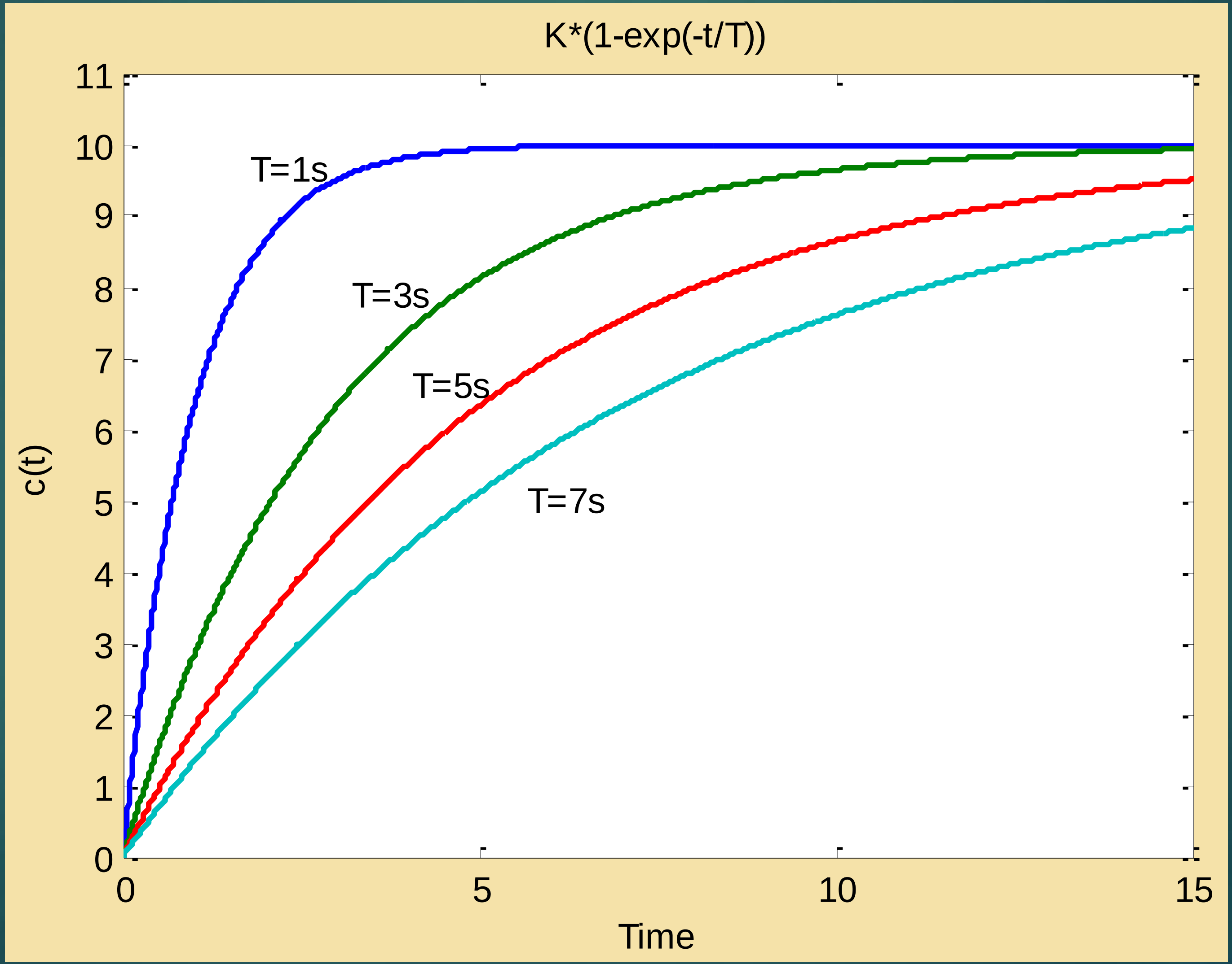
- When t=T (time constant)

$$c(t) = \left(1 - e^{-1}\right) = 0.632$$

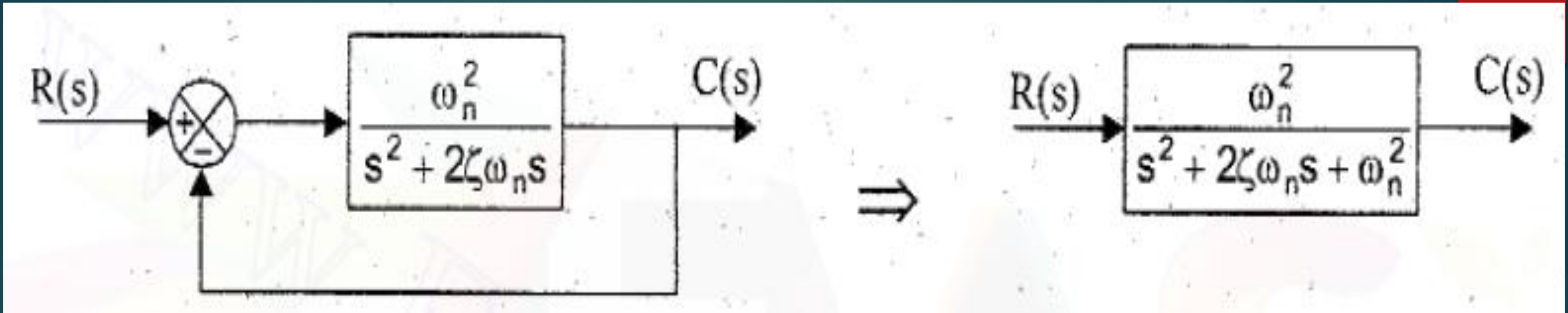
Step Response of 1st Order System

- **T=1, 3, 5, 7 seconds**

$$c(t) = \left(1 - e^{-t/T}\right)$$



Step Response of 2nd Order System



ω_n → Un-damped natural frequency of oscillations

ζ → Damping factor

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

Damping - A reduction in the amplitude of an oscillation as a result of energy being drained from the system to overcome frictional or other resistive forces.

The ***damping ratio*** is defined as the ratio of the actual damping to the critical damping. The response c(t) of second order system depends on the value of damping ratio. Depending on the value of ζ, the system can be classified into the following four cases,

Case 1 : Undamped system, $\zeta = 0$

Case 2 : Under damped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Over damped system, $\zeta > 1$

The characteristics equation of the second order system is,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

It is a quadratic equation and the roots of this equation is given by,

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

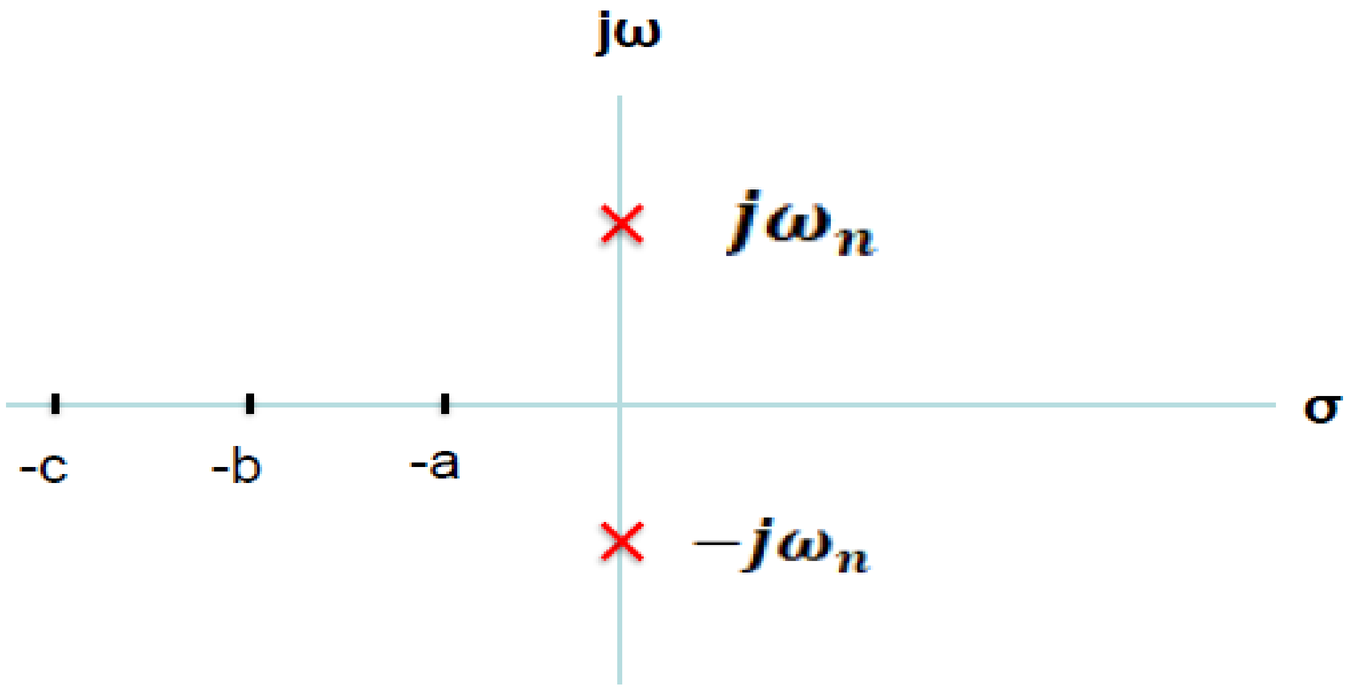
According the value of ξ , a second-order system can be classified into one of the four categories

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

When $\zeta = 0$, $s_1, s_2 = \pm j\omega_n$;

$\begin{cases} \text{roots are purely imaginary} \\ \text{and the system is undamped} \end{cases}$

Undamped ($\xi = 0$) -the system has *two imaginary poles*. $\pm j\omega_n$



The system response is oscillatory.

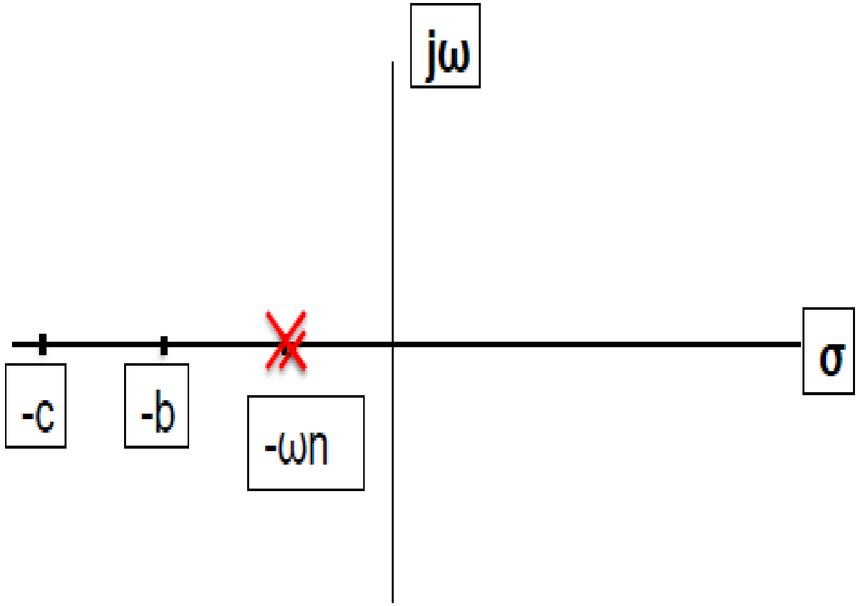
$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

When $\zeta = 1$, $s_1, s_2 = -\omega_n$

$\left\{\begin{array}{l}\text{roots are real and equal and} \\ \text{the system is critically damped}\end{array}\right.$

Critically damped ($\xi = 1$) - the system has two *real but equal poles*

The two poles are $-\omega_n$



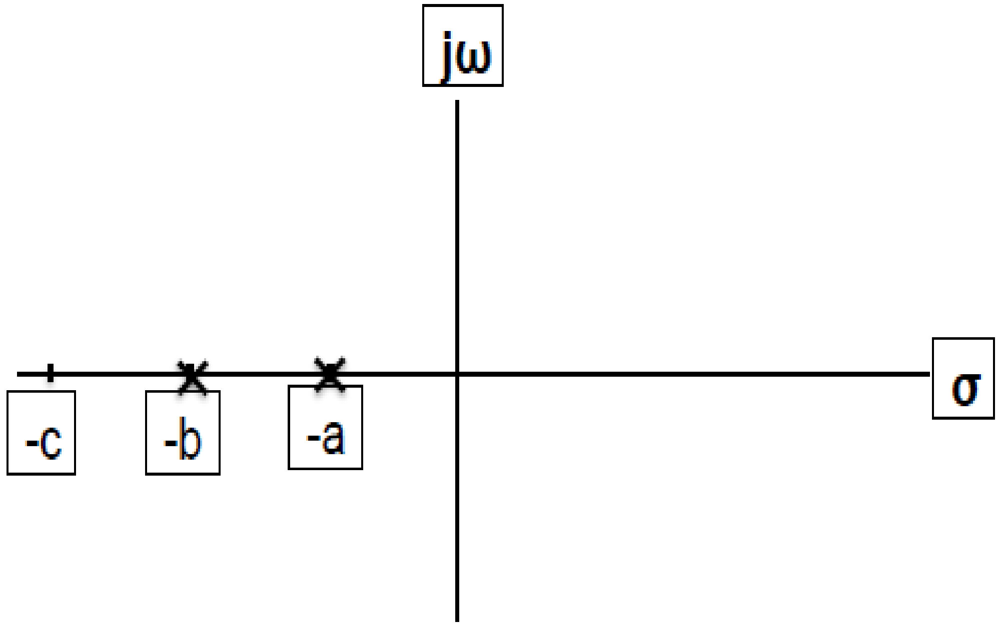
The response rises slowly and reaches the final value without any oscillations

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

When $\zeta > 1,\ s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

$\begin{cases} \text{roots are real and unequal and} \\ \text{the system is overdamped} \end{cases}$

Overdamped (ξ >1) - the system poles are *real and distinct*



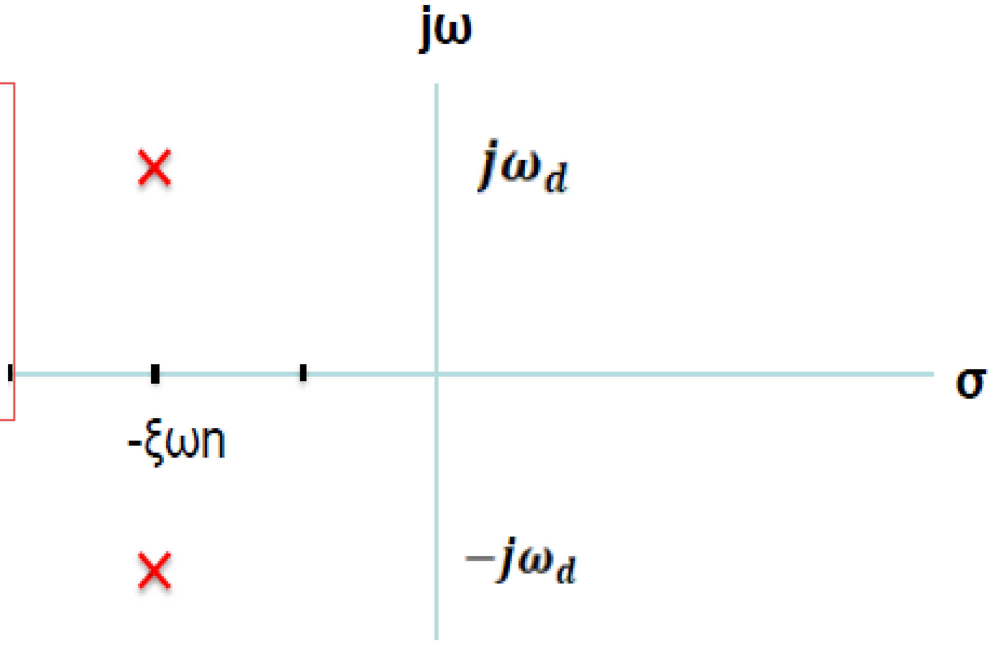
The response has no oscillations, but it require longer time to reach its steady state value

Underdamped $(0 < \xi < 1)$ - the system has a pair of complex conjugate poles $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

When $\zeta > 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$;

$= -\zeta\omega_n \pm j\omega_d$; $\begin{cases}\text{roots are complex conjugate}\\ \text{the system is underdamped}\end{cases}$

where, $\omega_d = \omega_n\sqrt{1-\zeta^2}$



The transient response is oscillatory.

RESPONSE OF UNDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For undamped system, z = 0.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

When the input is unit step, $r(t) = 1$ and $R(s) = \frac{1}{s}$.

When the input is unit step, $r(t) = 1$ and $R(s) = \frac{1}{s}$.

$\therefore$ The response in s-domain, $C(s) = R(s) \frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{1}{s} \frac{\omega_n^2}{s^2 + \omega_n^2}$

By partial fraction expansion,

$$C(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

$$\frac{w_n^2}{s(s^2 + w_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + w_n^2}$$

$$w_n^2 = A(s^2 + w_n^2) + (Bs + C)s$$

$$w_n^2 = As^2 + Aw_n^2 + Bs^2 + Cs$$

Compare the coefficients of s^2:

$0 = A + B$

$A = -B$

Compare the coefficients of s:

$0 = C \qquad \therefore C = 0$

Compare the coefficients of constant:

$w_n^2 = Aw_n^2$

$\therefore A = 1 \quad and \quad B = -1$

$$\frac{A}{s}+\frac{Bs+C}{s^2+w_n^2}=\frac{1}{s}+\frac{-s+0}{s^2+w_n^2}$$

$$C(s)=\frac{1}{s}-\frac{s}{s^2+w_n^2}$$

$$C(s)=\frac{1}{s}-\frac{s}{s^2+w_n^2}$$

we know that :

On taking inverse L.T. :

$$L^{-1}\left(\frac{1}{s}\right)=u(t)$$

$$c(t)=u(t)-\cos w_n t$$

$$L^{-1}\left(\frac{s}{s^2+w^2}\right)=\cos wt$$

Unit step response of II order system For Unit step input :

$$c(t)=1-\cos w_n t$$

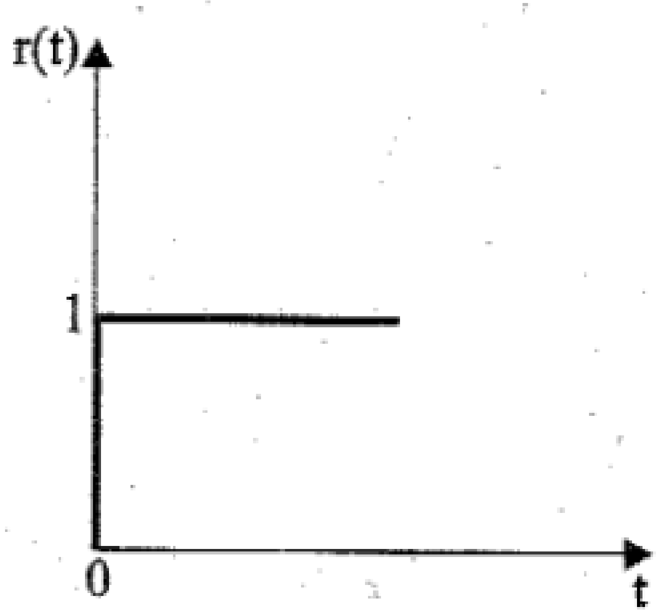


Fig 2.9.a : Input.

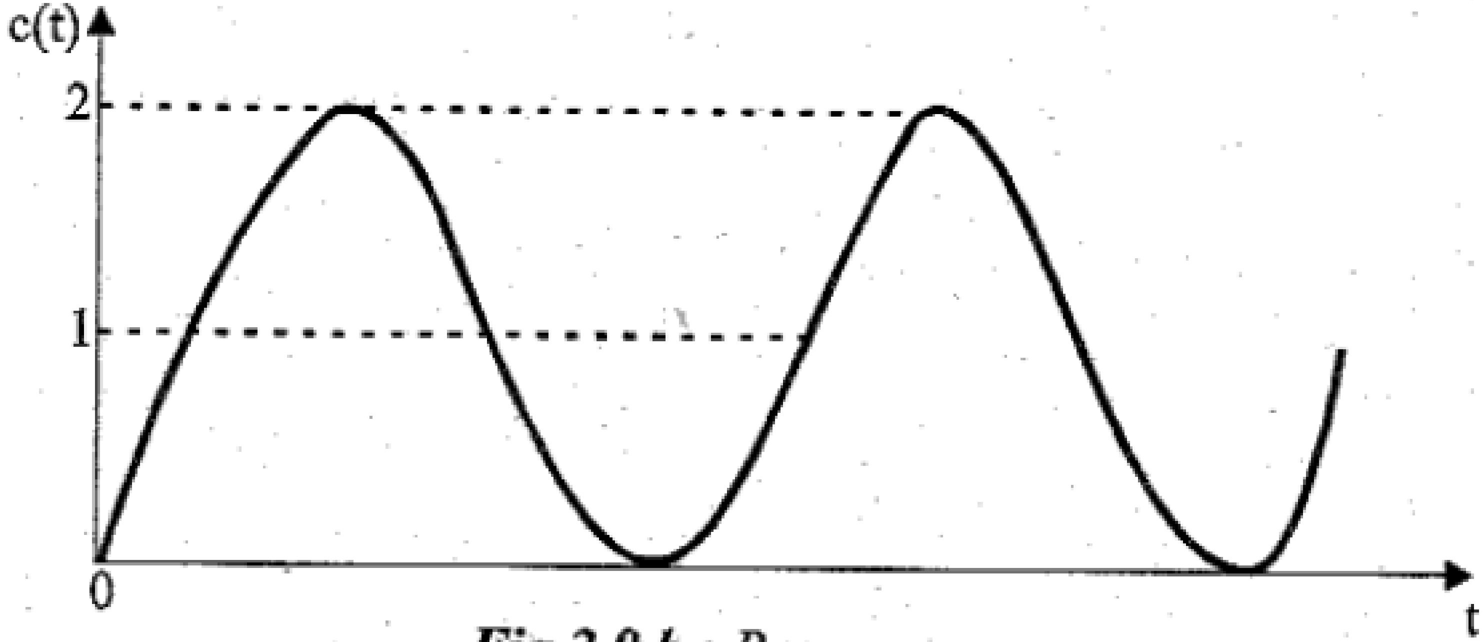


Fig 2.9.b : Response.

Fig 2.9 : *Response of undamped second order system for unit step input.*

> ***Note :*** *Every practical system has some amount of damping. Hence undamped system does not exist in practice.*

$\therefore$ For closed loop undamped second order system,

$$\text{Unit step response} = 1 - \cos \omega_n t$$

RESPONSE OF CRITICALLY DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For critical damping $\zeta = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

When input is unit step, $r(t) = 1$ and $R(s) = 1/s$.

$\therefore$ The response in s-domain,

$$C(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}\,\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{\omega_n^2}{s\,(s+\omega_n)^2}$$

By partial fraction expansion, we can write,

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$\frac{w_n^2}{s(s+w_n)^2} = \frac{A}{s} + \frac{B}{(s+w_n)^2} + \frac{C}{s+w_n}$$

$$w_n^2 = A(s+w_n)^2 + Bs + Cs(s+w_n)$$

$$Put\ \ s = 0$$

$$w_n^2 = Aw_n^2 \quad \therefore A = 1$$

$$Put\ \ s = -w_n$$

$$w_n^2 = B(-w_n) \quad \therefore B = -w_n$$

$$Put\ \ s = w_n$$

$$w_n^2 = A4w_n^2 + Bw_n + C2w_n^2$$

$$w_n^2 = 4w_n^2 - w_n^2 + C2w_n^2$$

$$w_n^2 = 3w_n^2 + C2w_n^2$$

$$w_n^2 = w_n^2[3+2C]$$

$$1 = 3+2C$$

$$2C = -2$$

$$C = -1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right\}$$

$$c(t) = 1 - \omega_n t\, e^{-\omega_n t} - e^{-\omega_n t}$$

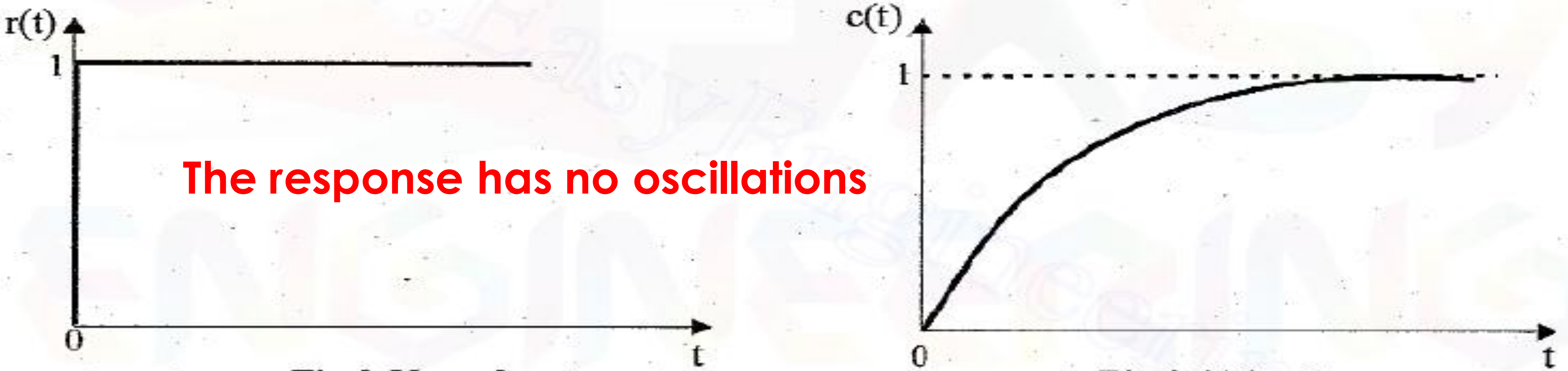
$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

We know that;

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$



The response has no oscillations

Fig 2.11.a : Input.

Fig 2.11.b : Response.

Fig 2.11 : Response of critically damped second order system for unit step input.

RESPONSE OF UNDERDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For underdamped system, $0 < \zeta < 1$ and roots of the denominator (characteristic equation) are complex conjugate.

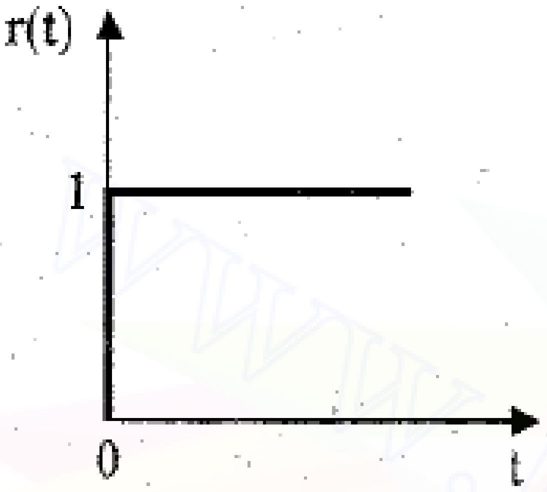


Fig 2.10.a : Input.

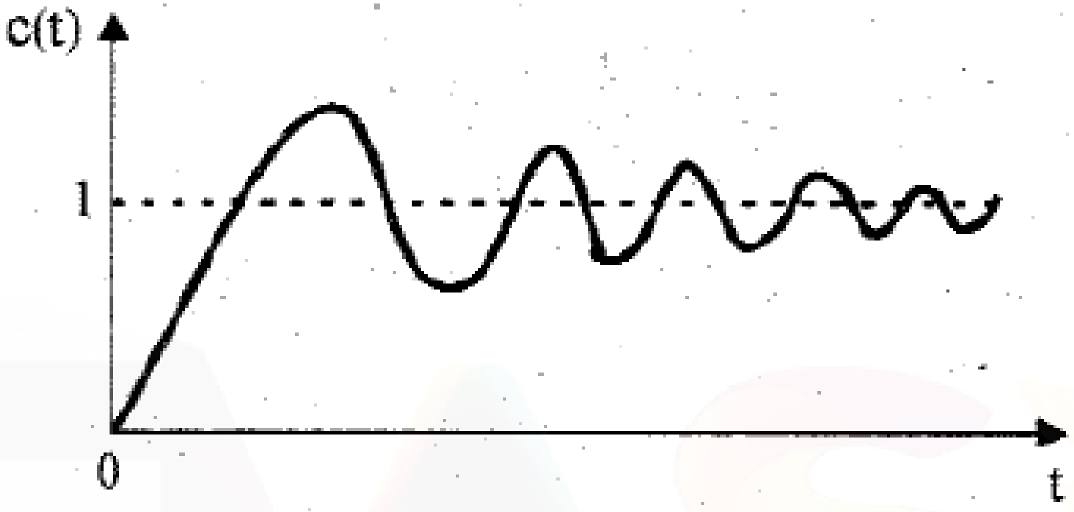


Fig 2.10.b : Response.

The response oscillates before setting to a final value. The oscillations depend on the damping ratio.

RESPONSE OF OVER DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For overdamped system $\zeta > 1$.

∴ For closed loop over damped second order system,

$$\text{Unit step response} = 1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \frac{1}{s_1}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)$$

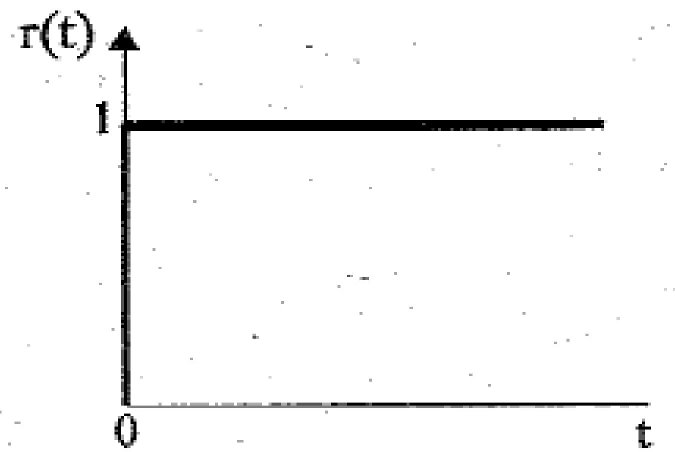


Fig 2.12.a : Input.

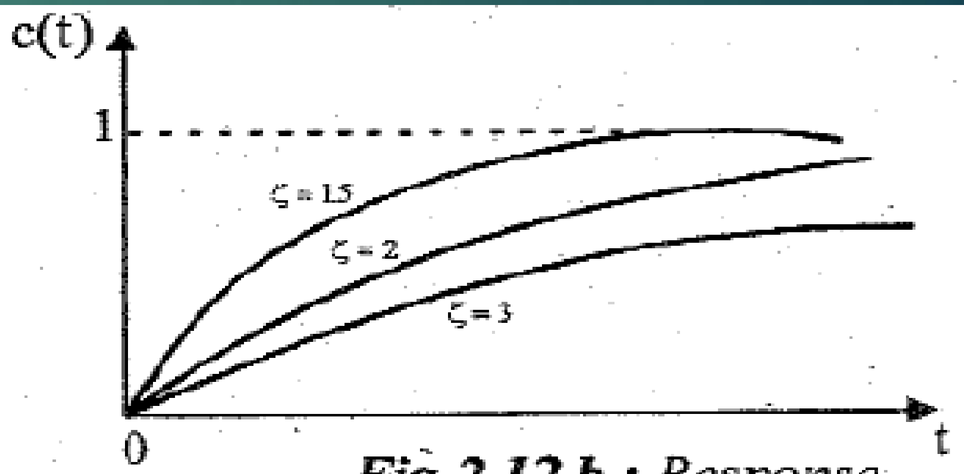


Fig 2.12.b : Response.

The response has no oscillations but it takes longer time to reach steady state value.

TIME DOMAIN SPECIFICATIONS

- The transient response of a practical control system often exhibits damped oscillations before reaching steady state.
- Actual output behavior according to the various time response specifications

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications.

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s

TIME DOMAIN SPECIFICATIONS

- The typical damped oscillatory response of a system is shown

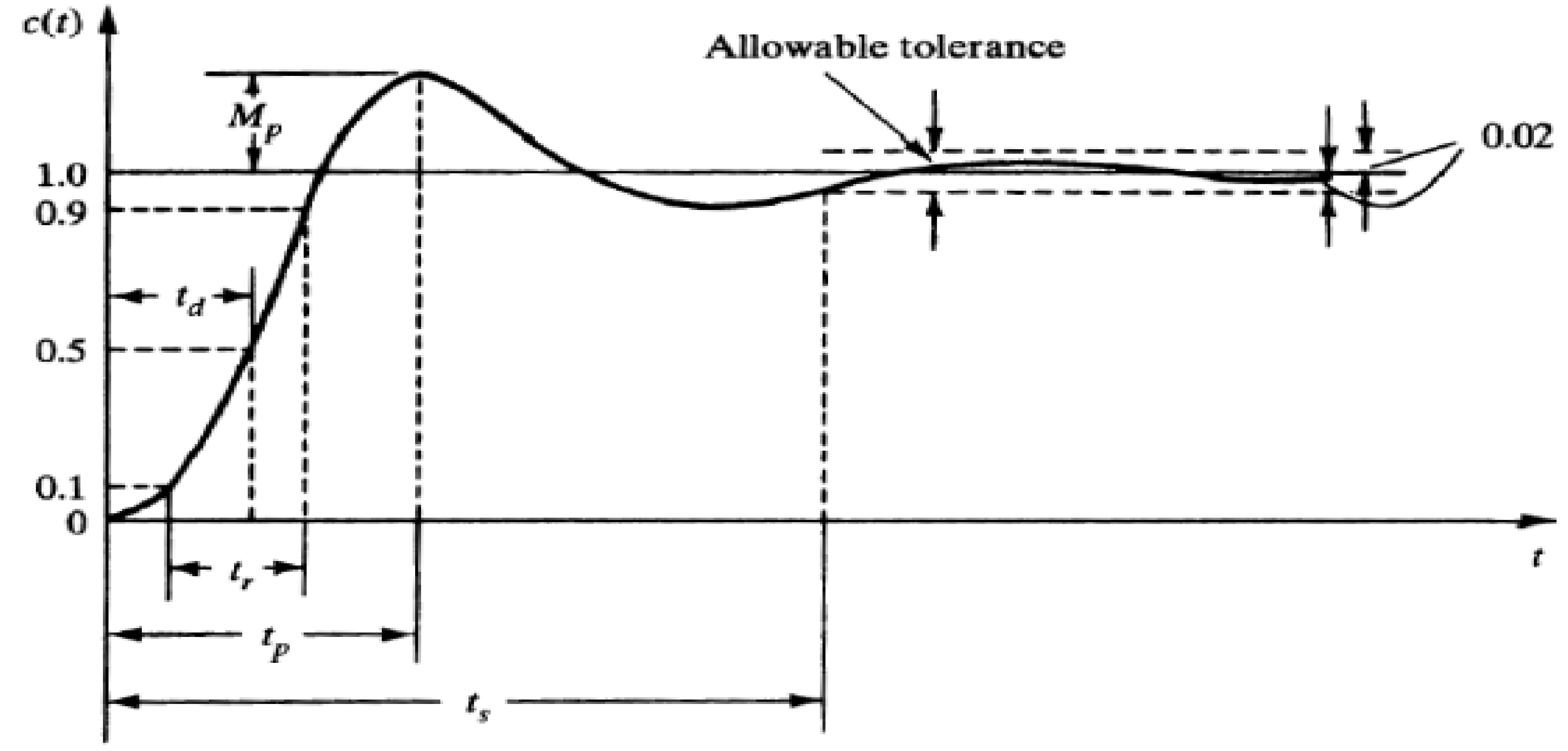


Figure 10–21 Transient-response specifications.

1) Delay Time, Td

It is the time taken for response to reach 50 % of the final value for the very first time

$$T_d \approx \frac{1+0.7\zeta}{\omega_n}$$

2) Rise Time, Tr

- It is the time taken for response to raise from 0 % to 100 % of the final value for the very first time. It is for undamped system.
- For over damped system. It is the time taken for response to raise from 10 % to 90 % .
- For critically damped system. It is the time taken for response to raise from 5 % to 95 %

$$\therefore \text{ Rise Time, } t_r = \frac{\pi - \theta}{\omega_d}$$

$$\therefore \text{ Rise time, } t_r = \frac{\pi - \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \text{ in sec}$$

3) Peak Time, Tp

It is the time required for the response to reach its peak value for the very first time.

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

4) Peak Overshoot, Mp

It is defined as the ratio of the maximum peak value to the final value

Let, $c(\infty)$ = Final value of $c(t)$.

$c(t_p)$ = Maximum value of $c(t)$.

Now, Peak overshoot, $M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$M_p\% = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$$

5) Settling Time, Ts

Time required for the response to decrease and stay within specified percentage of its final value (Usual tolerable error : 2 % or 5 % of final value)

$$T_s = \frac{4}{\zeta\omega_n} \qquad (2\%\ \text{Criterion})$$

$$T_s = \frac{3}{\zeta\omega_n} \qquad (5\%\ \text{Criterion})$$

Example #1

Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the input is unit step.

SOLUTION

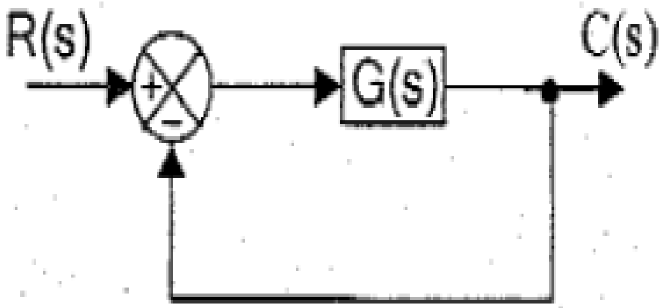


Fig 1 : Closed loop system.

The closed loop system is shown in fig 1.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{4}{s(s+5)}}{1+\frac{4}{s(s+5)}} = \frac{\frac{4}{s(s+5)}}{\frac{s(s+5)+4}{s(s+5)}} = \frac{4}{s(s+5)+4} = \frac{4}{s^2+5s+4} = \frac{4}{(s+4)(s+1)}$$

The response in s-domain, $C(s) = R(s) \frac{4}{(s+1)(s+4)}$

Since the input is unit step, $R(s) = \frac{1}{s}$; $\therefore C(s) = \frac{4}{s(s+1)(s+4)}$

By partial fraction expansion, we can write,

$$C(s) = \frac{4}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$\frac{4}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$4 = A(s+1)(s+4) + B\,s(s+4) + C\,s(s+1)$$

$Put\ \ s = 0$

$4 = 4A \quad \therefore\ A = 1$

$Put\ \ s = -1$

$4 = -3B \quad \therefore\ B = -4/3$

$Put\ \ s = -4$

$4 = C(-4)(-3) \quad \therefore\ C = 1/3$

$$C(s) = \frac{4}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

The time domain response c(t) is obtained by taking inverse Laplace transform of C(s).

$$\text{Response in time domain, } c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{4}{3}\,\frac{1}{s+1} + \frac{1}{3}\,\frac{1}{s+4}\right\}$$

$$= 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t} = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right].$$

RESULT

Response of unity feedback system, $c(t) = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$

Example #2

The unity feedback system is characterized by an open loop transfer function G(s) =K/s (s +10). Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine peak overshoot and time at peak overshoot for a unit step input.

SOLUTION

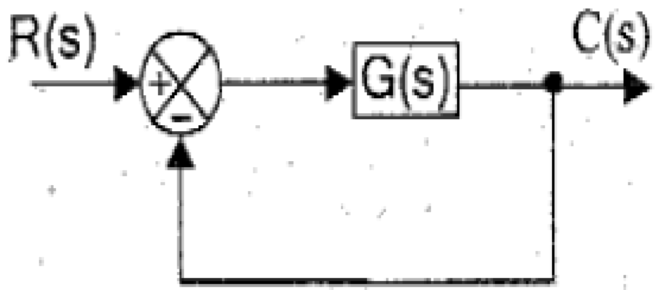


Fig 1 : Unity feedback system.

The unity feedback system is shown in fig 1.

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, G(s) = K/s (s +10)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s\,(s+10)+K} = \frac{K}{s^2+10s+K}$$

The value of K can be evaluated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{s^2 + 10s + K}$$

On comparing we get,

$\omega_n^2 = K$	$2\zeta\omega_n = 10$	$K = 100$
$\therefore \omega_n = \sqrt{K}$	Put $\zeta = 0.5$ and $\omega_n = \sqrt{K}$	$\omega_n = 10$ rad / sec
	$\therefore 2 \times 0.5 \times \sqrt{K} = 10$	
	$\sqrt{K} = 10$	

The value of gain, K =100.

Percentage peak overshoot, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 0.163 \times 100 = 16.3\%$$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.363 \text{ sec}$

RESULT

The value of gain,	K	=	100
Percentage peak overshoot,	$\%M_p$	=	16.3%
Peak time,	t_p	=	0.363 sec.

Example #3

A positional control system with velocity feedback is shown in fig 1. What is the response c(t) to the unit step input. Given hat $\zeta = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.

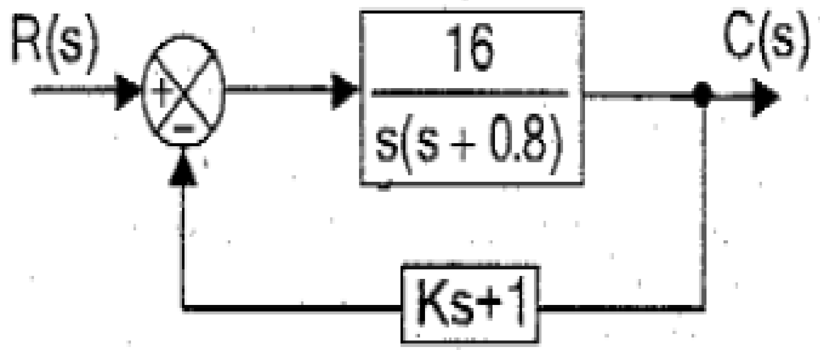


Fig 1

SOLUTION

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$

Given that G(s) = 16/s(s + 0.8) and H(s) = Ks + 1

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{16}{s\,(s+0.8)}}{1+\frac{16}{s\,(s+0.8)}(Ks+1)} = \frac{16}{s\,(s+0.8)+16(Ks+1)}$$

$$= \frac{16}{s^2+0.8s+16Ks+16} = \frac{16}{s^2+(0.8+16K)\,s+16}$$

The values of K and ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{16}{s^2 + (0.8 + 16K)\, s + 16}$$

On comparing we get.

$$\omega_n^2 = 16 \qquad 0.8 + 16K = 2\zeta\omega_n$$

$$\therefore \omega_n = 4 \text{ rad/sec} \qquad \therefore K = \frac{2\zeta\omega_n - 0.8}{16} = \frac{2 \times 0.5 \times 4 - 0.8}{16} = 0.2$$

$$\therefore \frac{C(s)}{R(s)} = \frac{16}{s^2 + (0.8 + 16 \times 0.2)\, s + 16} = \frac{16}{s^2 + 4s + 16}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{16}{s^2 + (0.8 + 16 \times 0.2)\, s + 16} = \frac{16}{s^2 + 4s + 16}$$

Given that the damping ratio, $\zeta = 0.5$. Hence the system is underdamped and so the response of the system will have damped oscillations. The roots of characteristic polynomial will be complex conjugate.

The response in s - domain, $C(s) = R(s)\, \frac{16}{s^2 + 4s + 16}$

For unit step input, $R(s) = 1/s$.

$$\therefore C(s) = \frac{1}{s}\, \frac{16}{s^2 + 4s + 16} = \frac{16}{s\,(s^2 + 4s + 16)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{16}{s(s^2+4s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

$$\frac{16}{s(s^2+4s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

$$16 = A(s^2+4s+16) + (Bs+C)s$$

$$16 = As^2 + 4As + 16A + Bs^2 + Cs$$

Compare the coeff of s^2 :

$$0 = A + B \quad \therefore A = -B$$

Compare the coeff of s :

$$0 = 4A + C \rightarrow (1)$$

Compare the constant coeff :

$$16 = 16A \quad \therefore A = 1 \ \& \ B = -1$$

Substitute the above values in Eqn.1

$$0 = 4 + C \quad \therefore C = -4$$

$$\frac{16}{s\,(s^2+4s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

$$\therefore\ C(s) = \frac{1}{s} + \frac{-s-4}{s^2+4s+16} = \frac{1}{s} - \frac{s+4}{s^2+4s+4+12}$$

$$= \frac{1}{s} - \frac{s+2+2}{(s+2)^2+12} = \frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\ \frac{\sqrt{12}}{(s+2)^2+12}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\ \frac{\sqrt{12}}{(s+2)^2+12}\right\}$$

$$= 1 - e^{-2t}\cos\sqrt{12}\,t - \frac{2}{2\sqrt{3}}\ e^{-2t}\sin\sqrt{12}\,t$$

$$= 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$$

$$\left.\begin{array}{l}\text{Damped frequency}\\ \text{of oscillation}\end{array}\right\}\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464 \text{ rad / sec}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907 \text{ sec}$$

$$\left.\begin{array}{r}\%\text{ Maximum}\\ \text{overshoot}\end{array}\right\}\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\times\pi}{\sqrt{1-0.5^2}}} \times 100 = 0.163 \times 100 = 16.3\%$$

$$Settling\ \ time\ t_s(5\%\ error) = \frac{3}{\varsigma\omega_n} = \frac{3}{0.5\times 4} = 1.5\,\text{sec}$$

$$Settling\ \ time\ t_s(2\%\ error) = \frac{4}{\varsigma\omega_n} = \frac{4}{0.5\times 4} = 2\,\text{sec}$$

Exercise Problem #1

A unity feedback control system has an open loop transfer function G(s)=6/s(s+5). Obtain the response of the system and find the rise time, peak time, delay time, maximum peak overshoot and settling time for a unit step input.

Solution:

Soln:

Tr. fn.,
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\frac{6}{s(s+5)}}{1+\frac{6}{s(s+5)}\cdot 1} \quad \because H(s)=1$$

$$= \frac{\frac{6}{s(s+5)}}{\frac{s(s+5)+6}{s(s+5)}} = \frac{6}{s(s+5)+6}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{6}{s^2+5s+6}} = \frac{6}{(s+3)(s+2)}$$

$$\frac{C(s)}{R(s)} = \frac{6}{s^2+5s+6} = \frac{6}{(s+3)(s+2)}$$

For unit step input, $R(s) = 1/s$

$$\therefore C(s) = R(s) \cdot \frac{6}{(s+3)(s+2)}$$

$$C(s) = \frac{6}{s(s+3)(s+2)}$$

To find the response, $c(t)$:

Apply Partial Fraction:

$$C(s) = \frac{6}{s(s+3)(s+2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$\frac{6}{s(s+3)(s+2)} = \frac{A(s+3)(s+2) + B \cdot s(s+2) + C(s)(s+3)}{s(s+3)(s+2)}$$

$$6 = A(s+3)(s+2) + Bs(s+2) + Cs(s+3)$$

put $s = 0$

$$6 = 6A$$

$$\therefore \boxed{A = 1}$$

Put $s = -3$

$$6 = 3B$$

$$\therefore \boxed{B = 2}$$

Put $s = -2$

$$6 = -2C$$

$$\boxed{C = -3}$$

$$\therefore C(s) = \frac{1}{s} + \frac{2}{s+3} - \frac{3}{s+2}$$

Take inv. Laplace transform:

$$\boxed{c(t) = 1 + 2e^{-3t} - 3e^{-2t}}$$

We know that; For II order System

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{6}{(s+3)(s+2)} = \frac{6}{s^2+5s+6}$$

$\therefore$ $\omega_n^2 = 6$

$\omega_n = \sqrt{6}$

$\boxed{\omega_n = 2.45}$

$2\zeta\omega_n = 5$

$\zeta = \frac{5}{2 \times \omega_n} = \frac{5}{2 \times \sqrt{6}}$

$\boxed{\zeta = 1.02}$

(i) Delay time, t_d:

$$t_d = \frac{1 + 0.7\zeta}{\omega_n} = \frac{1 + 0.7 \times 1.02}{2.45} = \frac{1 + 0.714}{2.45}$$

$\boxed{t_d = 0.69 \text{ sec}}$

(ii) Rise time, t_r:

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$t_r = \infty$$

$$\theta = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$= \tan^{-1} \frac{\sqrt{1-(1.02)^2}}{1.02}$$

$$\theta = \infty$$

$$\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$$

$$= 2.45 \sqrt{1-\zeta^2}$$

$$= \infty$$

(iii) Peak time, t_p:

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\infty} = \infty .$$

(iv) Peak overshoot, ∵ not possible

(v) settling time, $t_s(5\%) = \frac{3}{\zeta \omega_n} = \frac{3}{1.02 \times 2.45}$

$t_s(5\%) = 1.20$ sec

$t_s(2\%) = \frac{4}{\zeta \times \omega_n} = 1.60$ sec.

Steady State Error

Steady-state error is defined as the difference between the input and the output of a system in the limit as time goes to infinity (i.e. when the response has reached steady state).

The parameter that is important in this is the steady state error (Ess).

Steady state error is error at t→∞.

$$E_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

The steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ sE(s) = \underset{s\to 0}{Lt}\ \frac{sR(s)}{1+G(s)\,H(s)}$

- Any physical control system inherently suffers steady-state error in response to certain types of inputs.
- A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.
- Whether a given system will exhibit steady-state error for a given type of input depends on the type of open-loop transfer function of the system.

- The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

TYPE NUMBER OF CONTROL SYSTEMS

The type number is specified for loop transfer function G(s) H(s). The number of poles of the loop transfer function lying at the origin decides the type number of the system. In general, if **N** is the number of poles at the origin then the type number is **N.**

The loop transfer function can be expressed as a ratio of two polynomials in s.

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\ldots\ldots\ldots\ldots}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\ldots\ldots\ldots}$$

where, z_1, z_2, z_3, ... are zeros of transfer function

p_1, p_2, p_3, ... are poles of transfer function

K = Constant

N = Number of poles at the origin

$$G(s)H(s) = \frac{s+7}{s(s+1)(s+4)}$$

$$G(s)H(s) = \frac{6}{s^2(s+1)(s+4)}$$

- As the type number is increased, accuracy is improved.
- However, increasing the type number aggravates the stability problem.
- A compromise between steady-state accuracy and relative stability is always necessary.

Type-0 system will have a constant steady state error when the input is step signal. Type-1 system will have a constant steady state error when the input is ramp signal or velocity signal. Type-2 system will have a constant steady state error when the input is parabolic signal or acceleration signal. For the three

Static Error Constants

- The static error constants are figures of merit of control systems. The higher the constants, the smaller the steady-state error.

- In a given system, the output may be the position, velocity, pressure, temperature, or the like.

- Therefore, in what follows, we shall call the output "position," the rate of change of the output "velocity," and so on.

- This means that in a temperature control system "position" represents the output temperature, "velocity" represents the rate of change of the output temperature, and so on.

The steady state error associated with the any one of the following constants:

Positional error constant, $K_p = \underset{s \to 0}{Lt}\ G(s)\,H(s)$

Velocity error constant, $K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s)$

Acceleration error constant, $K_a = \underset{s \to 0}{Lt}\ s^2 G(s)\,H(s)$

The K_p, K_v and K_a are in general called static error constants.

Static Position Error Constant (K_p)

- The steady-state error of the system for a unit-step input is

Steady state error, $e_{ss} = \underset{s\to 0}{Lt} \dfrac{sR(s)}{1+G(s)H(s)}$

When the input is unit step, $R(s) = 1/s$

$$\therefore e_{ss} = \underset{s\to 0}{Lt} \frac{s\frac{1}{s}}{1+G(s)H(s)} = \underset{s\to 0}{Lt} \frac{1}{1+G(s)H(s)} = \frac{1}{1+\underset{s\to 0}{Lt}\, G(s)H(s)} = \frac{1}{1+K_p}$$

where, $K_p = \underset{s\to 0}{Lt}\, G(s)H(s)$

The constant K_p is called ***positional error constant.***

Static Velocity Error Constant (K_v)

- The steady-state error of the system for a unit-ramp input is

Steady state error, $e_{ss} = \underset{s \to 0}{Lt} \dfrac{sR(s)}{1+G(s)H(s)}$

When the input is unit ramp, $R(s) = \dfrac{1}{s^2}$

$$\therefore e_{ss} = \underset{s \to 0}{Lt} \frac{s\frac{1}{s^2}}{1+G(s)H(s)} = \underset{s \to 0}{Lt} \frac{1}{s+sG(s)H(s)} = \frac{1}{\underset{s \to 0}{Lt}\ sG(s)H(s)} = \frac{1}{K_v}$$

$$\text{where, } K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s)$$

The constant K_v is called ***velocity error constant.***

Static Acceleration Error Constant (K_a)

- The steady-state error of the system for parabolic input is

Steady state error, $e_{ss} = \underset{s\to 0}{Lt} \frac{sR(s)}{1+G(s)H(s)}$

When the input is unit parabola, $R(s) = \frac{1}{s^3}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt} \frac{s\frac{1}{s^3}}{1+G(s)H(s)} = \underset{s\to 0}{Lt} \frac{1}{s^2+s^2 G(s)H(s)} = \frac{1}{\underset{s\to 0}{Lt}\ s^2G(s)H(s)} = \frac{1}{K_a}$$

where, $K_a = \underset{s\to 0}{Lt}\ s^2G(s)H(s)$

The constant K_a is called ***acceleration error constant.***

TABLE-2.2 : Static Error Constant for Various Type Number of Systems

Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

TABLE-2.3 : Steady State Error for Various Types of Inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

Example (evaluation of Static Error Constants)

$$G(s) = \frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}$$

$$K_p = \lim_{s \to 0} G(s)$$

$$K_p = \lim_{s \to 0} \left(\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_p = \infty$$

$$K_v = \lim_{s \to 0} sG(s)$$

$$K_v = \lim_{s \to 0} \left(\frac{100s(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_v = \infty$$

$$K_a = \lim_{s \to 0} s^2 G(s)$$

$$K_a = \lim_{s \to 0} \left(\frac{100s^2(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_a = \left(\frac{100(0+2)(0+5)}{(0+8)(0+12)} \right) = 10.4$$

Example (Steady Sate Errors)

$$K_p = \infty \qquad K_v = \infty \qquad K_a = 10.4$$

$$e_{ss} = \frac{1}{1 + K_p} = 0$$

$$e_{ss} = \frac{1}{K_v} = 0$$

$$e_{ss} = \frac{1}{K_a} = 0.09$$

Exercise problem:

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$; b) $G(s) = \dfrac{10}{(s+2)(s+3)}$;

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$

The steady state error with unit velocity input, $e_{ss} = \dfrac{1}{K_v}$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s\to 0}{Lt}\ s\,G(s)$

$$= \underset{s\to 0}{Lt}\ s\,\frac{20(s+2)}{s(s+1)(s+3)} = \frac{20\times 2}{1\times 3} = \frac{40}{3}$$

Steady state error, $e_{ss} = \dfrac{1}{K_v} = \dfrac{3}{40} = 0.075$

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

b) $G(s) = \dfrac{10}{(s+2)(s+3)}$

The steady state error with unit step input, $e_{ss} = \dfrac{1}{1+K_p}$

Position error constant, $K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \dfrac{10}{(s+2)(s+3)} = \dfrac{10}{2\times 3} = \dfrac{5}{3}$

Steady state error, $e_{ss} = \dfrac{1}{1+K_p} = \dfrac{1}{1+\dfrac{5}{3}} = \dfrac{3}{3+5} = \dfrac{3}{8} = 0.375$

Innovative Method: Flipped class room – P, PI, PID controllers

1. Controllers and types of controllers

2. Advantages and disadvantages of controllers

3. Real time application of controllers

Innovative Method: Flipped class room – P, PI, PID controllers

https://www.youtube.com/watch?v=9H3-xM7Oj5c

https://www.youtube.com/watch?v=7oZgcAtgG9O

https://www.youtube.com/watch?v=OYPXiC7QjUM

https://www.youtube.com/watch?v=UROhOmjaHpO

https://www.youtube.com/watch?v=CQA8miYlY1Q

https://www.youtube.com/watch?v=sFqFrmMJ-sg

https://www.youtube.com/watch?v=JFTJ2SS4xyA

https://www.youtube.com/watch?v=d4sCLmTJbvM

Thank You